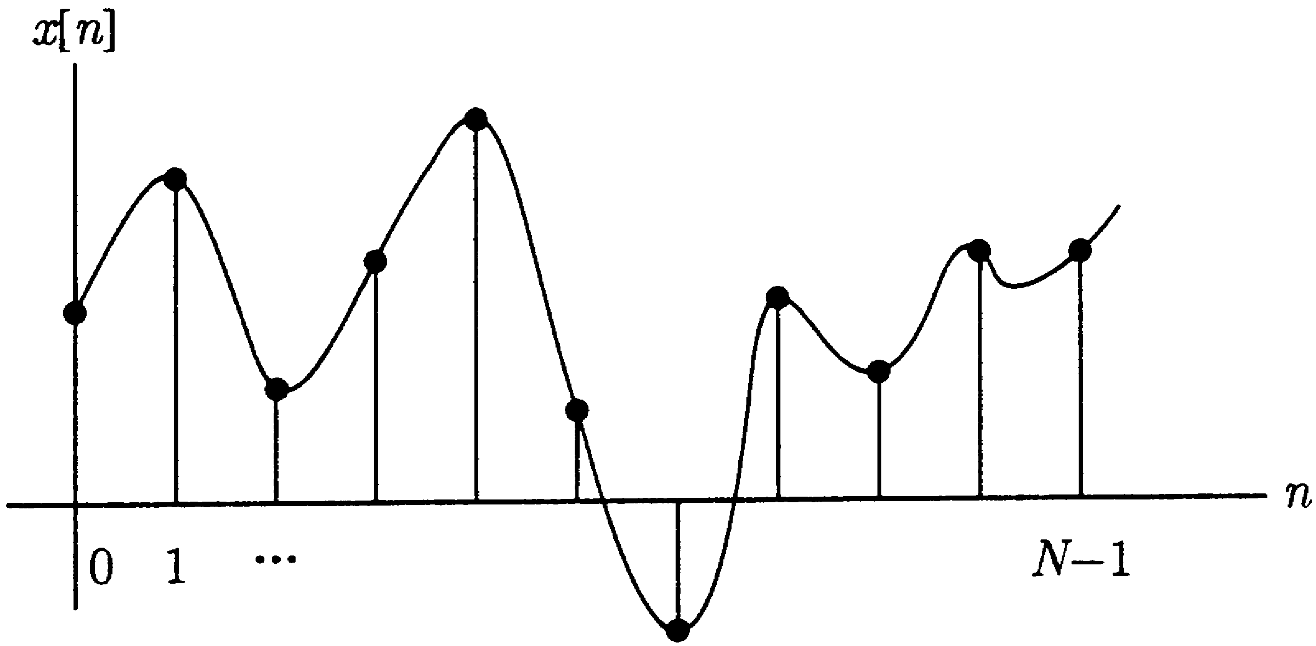
LINEAR ALGEBRA PRELIMINARIES

- Statistical characterization of random vectors
- Linear transformations of random vectors
- Reversal notation
- Correlation/covariance diagonalization
  - Eigenvector transformation
  - Triangular decomposition

REPRESENTATION OF A RANDOM SIGNAL AS A RANDOM VECTOR



$$\boldsymbol{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

EXPECTATION AND MOMENTS

EXPECTATION

$$E\{\boldsymbol{\psi}(\boldsymbol{x})\} = \int_{-\infty}^{\infty} \boldsymbol{\psi}(\mathbf{x}) f_{\boldsymbol{x}}(\mathbf{x}) d\mathbf{x}$$

$\boldsymbol{\psi}(\boldsymbol{x})$: any quantity (scalar, vector, matrix) depending on random vector $\boldsymbol{x}$

MEAN VECTOR

$$\mathbf{m}_{\boldsymbol{x}} = E\{\boldsymbol{x}\} = \int_{-\infty}^{\infty} \mathbf{x} f_{\boldsymbol{x}}(\mathbf{x}) d\mathbf{x}$$

EXPECTATION AND MOMENTS (cont'd.)

CORRELATION MATRIX

$$\mathrm{R}_{\boldsymbol{x}} = E\left\{\boldsymbol{x}\boldsymbol{x}^{*T}\right\} \;=\; E\left\{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}\begin{bmatrix} x_1^* & x_2^* & \cdots & x_N^* \end{bmatrix}\right\}$$

$$= \begin{bmatrix} E\left\{|x_1|^2\right\} & E\left\{x_1 x_2^*\right\} & \cdots & E\left\{x_1 x_N^*\right\} \\ E\left\{x_2 x_1^*\right\} & E\left\{|x_2|^2\right\} & \cdots & E\left\{x_2 x_N^*\right\} \\ \vdots & \vdots & & \vdots \\ E\left\{x_N x_1^*\right\} & E\left\{x_N x_2^*\right\} & \cdots & E\left\{|x_N|^2\right\} \end{bmatrix}$$

EXPECTATION AND MOMENTS (cont'd.)

COVARIANCE MATRIX

$$\mathbf{C}_{\boldsymbol{x}} = E\Big\{(\boldsymbol{x} - \mathbf{m}_{\boldsymbol{x}})(\boldsymbol{x} - \mathbf{m}_{\boldsymbol{x}})^{*T}\Big\}$$

Matrix elements are of the form $E\Big\{(x_i - \mathrm{m}_i)(x_j - \mathrm{m}_j)^*\Big\}$.

Diagonal elements are $E\Big\{|x_i - \mathrm{m}_i|^2\Big\}$ (variances of components).

RELATION

$$\mathbf{R}_{\boldsymbol{x}} = \mathbf{C}_{\boldsymbol{x}} + \mathbf{m}_{\boldsymbol{x}}\mathbf{m}_{\boldsymbol{x}}^{*T}$$

CORRELATION MATRIX PROPERTIES

1. Conjugate symmetry

$$\mathbf{R}_{\boldsymbol{x}} = \mathbf{R}_{\boldsymbol{x}}^{*T}$$

2. Positive semidefinite

$$\mathbf{a}^{*T}\mathbf{R}_{\boldsymbol{x}}\mathbf{a} \geq 0$$

   for any vector $\mathbf{a}$.

Identical properties hold for the covariance matrix.

CROSS-CORRELATION AND -COVARIANCE

DEFINITION

$$\mathrm{R}_{\boldsymbol{xy}} = E\left\{\boldsymbol{x}\boldsymbol{y}^{*T}\right\} \quad \text{and} \quad \mathrm{C}_{\boldsymbol{xy}} = E\left\{(\boldsymbol{x} - \mathbf{m}_{\boldsymbol{x}})(\boldsymbol{y} - \mathbf{m}_{\boldsymbol{y}})^{*T}\right\}$$

RELATION

$$\mathrm{R}_{\boldsymbol{xy}} = \mathrm{C}_{\boldsymbol{xy}} + \mathbf{m}_{\boldsymbol{x}}\mathbf{m}_{\boldsymbol{y}}^{*T}$$

These matrices have no specific properties except:

$$\mathrm{R}_{\boldsymbol{xy}} = \mathrm{R}_{\boldsymbol{yx}}^{*T} \quad \text{and} \quad \mathrm{C}_{\boldsymbol{xy}} = \mathrm{C}_{\boldsymbol{yx}}^{*T}$$

UNCORRELATED RANDOM VECTORS

Random vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are <u>uncorrelated</u> if

$$\mathbf{C}_{\boldsymbol{xy}} = E\left\{(\boldsymbol{x} - \mathbf{m}_{\boldsymbol{x}})(\boldsymbol{y} - \mathbf{m}_{\boldsymbol{y}})^{*T}\right\} = [\mathbf{0}]$$

This is equivalent to the statement $\mathbf{R}_{\boldsymbol{xy}} = \mathbf{m}_{\boldsymbol{x}}\mathbf{m}_{\boldsymbol{y}}^{*T}$ or

$$E\left\{\boldsymbol{x}\boldsymbol{y}^{*T}\right\} = E\{\boldsymbol{x}\}\, E\left\{\boldsymbol{y}^{*T}\right\}$$

Random vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are <u>orthogonal</u> if

$$\mathbf{R}_{\boldsymbol{xy}} = E\left\{\boldsymbol{x}\boldsymbol{y}^{*T}\right\} = [\mathbf{0}]$$

LINEAR TRANSFORMATIONS

$$\boldsymbol{y} = \mathbf{A}\boldsymbol{x}$$

MEAN VECTOR

$$E\{\boldsymbol{y}\} = E\{\mathbf{A}\boldsymbol{x}\} = \mathbf{A}E\{\boldsymbol{x}\} \quad \text{or } \ldots \quad \boxed{\mathbf{m}_{\boldsymbol{y}} = \mathbf{A}\mathbf{m}_{\boldsymbol{x}}}$$

CORRELATION MATRIX

$$E\left\{\boldsymbol{y}\boldsymbol{y}^{*T}\right\} = E\left\{(\mathbf{A}\boldsymbol{x})(\mathbf{A}\boldsymbol{x})^{*T}\right\} = \mathbf{A}E\left\{\boldsymbol{x}\boldsymbol{x}^{*T}\right\}\mathbf{A}^{*T}$$

$$\text{or } \ldots \quad \boxed{\mathbf{R}_{\boldsymbol{y}} = \mathbf{A}\mathbf{R}_{\boldsymbol{x}}\mathbf{A}^{*T}}$$

COVARIANCE MATRIX

$$\text{correspondingly } \ldots \quad \boxed{\mathbf{C}_{\boldsymbol{y}} = \mathbf{A}\mathbf{C}_{\boldsymbol{x}}\mathbf{A}^{*T}}$$

VECTOR AND MATRIX NORMS

EUCLIDEAN NORM OF A VECTOR

$$\|\boldsymbol{x}\| \stackrel{\text{def}}{=} \left(\sum_{k=1}^{N} |x_k|^2\right)^{\frac{1}{2}} = (\boldsymbol{x}^{*T}\boldsymbol{x})^{\frac{1}{2}}$$

EUCLIDEAN NORM OF A MATRIX

$$\|\mathbf{A}\| \stackrel{\text{def}}{=} \max_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\|$$

FROBENIUS NORM OF A MATRIX

$$\|\mathbf{A}\|_F \stackrel{\text{def}}{=} \left(\sum_{i=1}^{M}\sum_{j=1}^{N} |a_{ij}|^2\right)^{\frac{1}{2}} = \left(\text{tr } \mathbf{A}\mathbf{A}^{*T}\right)^{\frac{1}{2}}$$

REVERSAL OPERATION

VECTOR

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

REVERSAL OF VECTOR

$$\tilde{\boldsymbol{x}} = \begin{bmatrix} x_N \\ x_{N-1} \\ \vdots \\ x_1 \end{bmatrix}$$

REVERSAL OPERATION (cont'd.)

MATRIX

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

REVERSAL OF MATRIX

$$\tilde{\mathbf{A}} = \begin{bmatrix} a_{33} & a_{32} & a_{31} \\ a_{23} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{bmatrix}$$

REVERSAL IN A LINEAR TRANSFORMATION

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} y_3 \\ y_2 \\ y_1 \end{bmatrix} = \begin{bmatrix} a_{33} & a_{32} & a_{31} \\ a_{23} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}$$

$$\boldsymbol{y} = \mathbf{A}\boldsymbol{x} \qquad \Longleftrightarrow \qquad \tilde{\boldsymbol{y}} = \tilde{\mathbf{A}}\tilde{\boldsymbol{x}}$$

PROPERTIES OF REVERSAL

	Quantity	Reversal
Matrix product	$\mathbf{AB}$	$\tilde{\mathbf{A}}\tilde{\mathbf{B}}$
Matrix inverse	$\mathbf{A}^{-1}$	$(\tilde{\mathbf{A}})^{-1}$
Matrix conjugate	$\mathbf{A}^*$	$(\tilde{\mathbf{A}})^*$
Matrix transpose	$\mathbf{A}^T$	$(\tilde{\mathbf{A}})^T$

MEAN, CORRELATION AND COVARIANCE FOR REVERSED RANDOM VECTORS

MEAN VECTOR

$$\mathbf{m}_{\tilde{\boldsymbol{x}}} = E\{\tilde{\boldsymbol{x}}\} = \tilde{\mathbf{m}}_{\boldsymbol{x}}$$

CORRELATION MATRIX

$$\mathbf{R}_{\tilde{\boldsymbol{x}}} = E\left\{\tilde{\boldsymbol{x}}\tilde{\boldsymbol{x}}^{*T}\right\} = \tilde{\mathbf{R}}_{\boldsymbol{x}}$$

COVARIANCE MATRIX

$$\mathbf{C}_{\tilde{\boldsymbol{x}}} = \tilde{\mathbf{C}}_{\boldsymbol{x}}$$

DIAGONALIZING THE CORRELATION MATRIX

TRANSFORMATION

$$\boldsymbol{x}' = \mathbf{A}\boldsymbol{x}$$

such that

$$E\left\{x'_k x'^*_l\right\} = 0 \qquad k \neq l$$

(Vector components are *orthogonal*.)

METHODS

- Eigenvector decomposition (unitary transformation)
- Triangular decomposition ("causal" transformation)

EIGENVECTOR TRANSFORMATION: BASICS

$$\mathbf{R}_{\boldsymbol{x}}\mathbf{e} = \lambda\mathbf{e} \quad \text{implies} \quad \mathbf{e}_l^{*T}\mathbf{R}_{\boldsymbol{x}}\mathbf{e}_k = \lambda_k\mathbf{e}_l^{*T}\mathbf{e}_k = \begin{cases} \lambda_k & \text{if} \quad l = k \\ 0 & \text{if} \quad l \neq k \end{cases}$$

The transformation

$$\boldsymbol{x}' = \mathbf{E}^{*T}\boldsymbol{x} = \begin{bmatrix} -- & \mathbf{e}_1^{*T} & -- \\ -- & \mathbf{e}_2^{*T} & -- \\ & \vdots & \\ -- & \mathbf{e}_N^{*T} & -- \end{bmatrix}\boldsymbol{x}$$

produces the correlation matrix:

$$\begin{bmatrix} -- & \mathbf{e}_1^{*T} & -- \\ -- & \mathbf{e}_2^{*T} & -- \\ & \vdots & \\ -- & \mathbf{e}_N^{*T} & -- \end{bmatrix}\mathbf{R}_{\boldsymbol{x}}\begin{bmatrix} | & | & & | \\ \mathbf{e}_1 & \mathbf{e}_1 & \cdots & \mathbf{e}_N \\ | & | & & | \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_N \end{bmatrix}$$

EIGENVECTOR TRANSFORMATION: SUMMARY

$$\boxed{\boldsymbol{x}' = \mathbf{E}^{*T}\boldsymbol{x} \qquad \Longleftrightarrow \qquad \mathrm{R}_{\boldsymbol{x}'} = \mathbf{E}^{*T}\mathrm{R}_{\boldsymbol{x}}\mathbf{E} = \boldsymbol{\Lambda}}$$

(transformation is unitary: $\mathbf{E}\mathbf{E}^{*T} = \mathbf{I} \quad \Longrightarrow \quad \mathbf{E}^{*T} = \mathbf{E}^{-1}$)

CORRELATION MATRIX REPRESENTATION

$$\mathrm{R}_{\boldsymbol{x}} = \mathbf{E}\boldsymbol{\Lambda}\mathbf{E}^{*T} \qquad \mathrm{R}_{\boldsymbol{x}}^{-1} = \mathbf{E}\boldsymbol{\Lambda}^{-1}\mathbf{E}^{*T}$$

OTHER RELATIONS

$$|\mathrm{R}_{\boldsymbol{x}}| = |\boldsymbol{\Lambda}| = \prod_{j=1}^{N} \lambda_j \qquad \mathrm{tr}\ \mathrm{R}_{\boldsymbol{x}} = \mathrm{tr}\ \boldsymbol{\Lambda} = \sum_{j=1}^{N} \lambda_j$$

SINGULAR VALUE DECOMPOSITION

$$\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{*T}$$

$$\mathbf{U} = \begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_K \\ | & | & & | \end{bmatrix} \qquad \mathbf{V} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_N \\ | & | & & | \end{bmatrix} \qquad (\mathbf{U}, \mathbf{V} \text{ unitary})$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{or} \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_K & 0 & \cdots & 0 \end{bmatrix}$$

$$\sigma_k \geq 0$$

USING SVD FOR EIGENVECTOR PROBLEMS

ESTIMATE FOR CORRELATION MATRIX

$$\hat{\mathrm{R}}_{\boldsymbol{x}} = \frac{1}{K}\mathbf{X}^{*T}\mathbf{X} \qquad (\mathbf{X} \text{ is } K \times N \text{ with } K > N)$$

RELATIONS

$$\hat{\mathbf{E}} = \mathbf{V}\,; \qquad \hat{\mathbf{e}}_k = \mathbf{v}_k, \quad \hat{\lambda}_k = \frac{1}{K}\sigma_k^2 \qquad k = 1, 2, \ldots, N$$

COVARIANCE DIAGONALIZATION

$$\boxed{\breve{\boldsymbol{x}} = \breve{\mathbf{E}}^{*T}\boldsymbol{x} \qquad \Longleftrightarrow \qquad \mathbf{C}_{\breve{\boldsymbol{x}}} = \breve{\mathbf{E}}^{*T}\mathbf{C}_{\boldsymbol{x}}\breve{\mathbf{E}} = \breve{\boldsymbol{\Lambda}}}$$

where

$$\breve{\mathbf{E}} = \begin{bmatrix} | & | & & | \\ \breve{\mathbf{e}}_1 & \breve{\mathbf{e}}_2 & \cdots & \breve{\mathbf{e}}_N \\ | & | & & | \end{bmatrix} \qquad \breve{\boldsymbol{\Lambda}} = \begin{bmatrix} \breve{\lambda}_1 & & & 0 \\ & \breve{\lambda}_2 & & \\ & & \ddots & \\ 0 & & & \breve{\lambda}_N \end{bmatrix}$$

Components $\breve{x}_k$ of $\breve{\boldsymbol{x}}$ are *uncorrelated.*

$\breve{\lambda}_k$ is the *variance* of $\breve{x}_k$.

MULTIVARIATE GAUSSIAN DENSITY

REAL RANDOM VECTOR

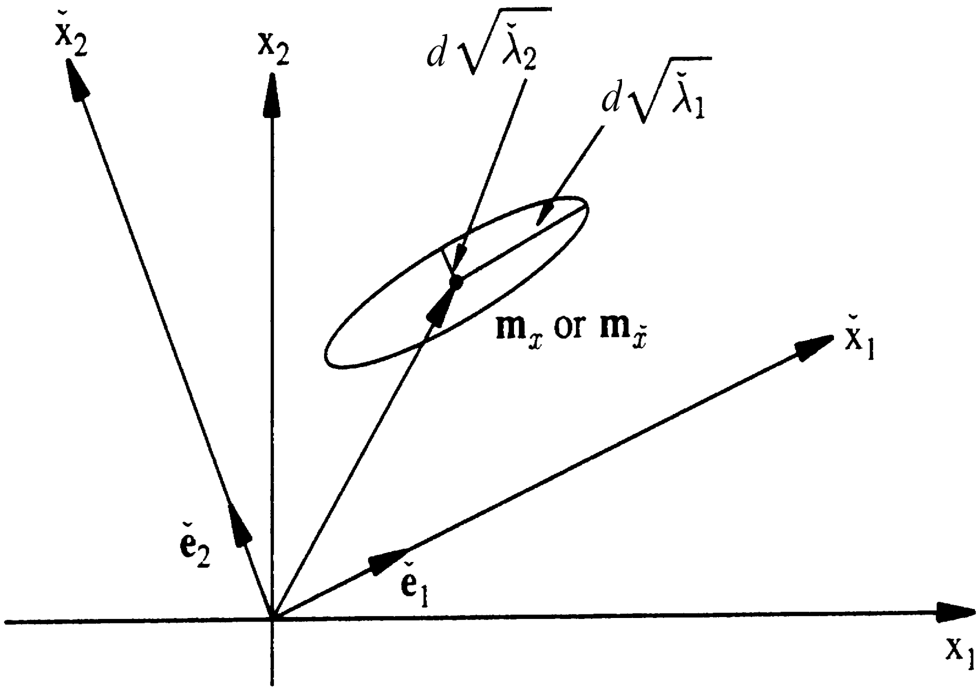
$$f_{\boldsymbol{x}}(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{N}{2}}|\mathbf{C}_{\boldsymbol{x}}|^{\frac{1}{2}}} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m}_{\boldsymbol{x}})^T \mathbf{C}_{\boldsymbol{x}}^{-1}(\mathbf{x}-\mathbf{m}_{\boldsymbol{x}})}$$

COMPLEX RANDOM VECTOR

$$f_{\boldsymbol{x}}(\mathbf{x}) = \frac{1}{\pi^N|\mathbf{C}_{\boldsymbol{x}}|} e^{-(\mathbf{x}-\mathbf{m}_{\boldsymbol{x}})^{*T} \mathbf{C}_{\boldsymbol{x}}^{-1}(\mathbf{x}-\mathbf{m}_{\boldsymbol{x}})}$$

CONCENTRATION ELLIPSOIDS
(CONTOURS OF THE GAUSSIAN DENSITY)

Contour defined by $(\mathbf{x}-\mathbf{m}_{\boldsymbol{x}})^{*T}\mathbf{C}_{\boldsymbol{x}}^{-1}(\mathbf{x}-\mathbf{m}_{\boldsymbol{x}}) = d^2$



- The transformation

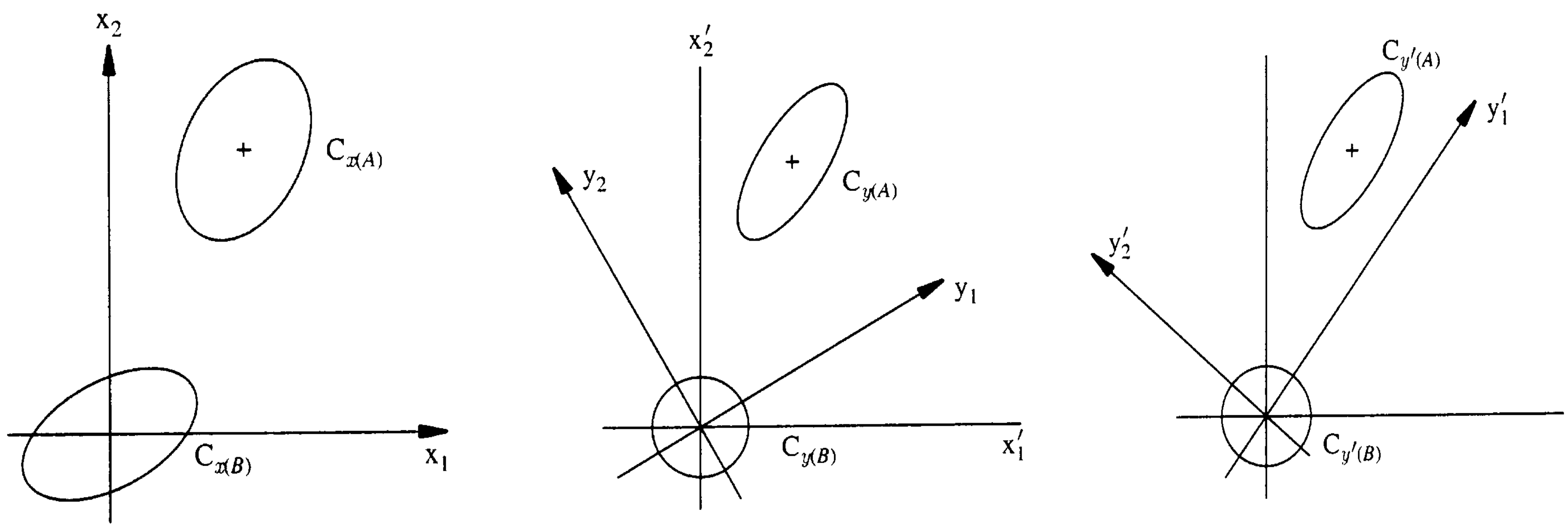
$$\check{\boldsymbol{x}} = \check{\mathbf{E}}^{*T}\boldsymbol{x}$$

  represents a rotation of coordinates.

- d is called the *Mahalanobis distance.*

SIMULTANEOUS DIAGONALIZATION

Covariance matrices $\mathbf{C}_{\boldsymbol{x}(A)}$ and $\mathbf{C}_{\boldsymbol{x}(B)}$ are transformed to diagonal forms $\mathbf{C}_{\boldsymbol{y}'(A)}$ and $\mathbf{C}_{\boldsymbol{y}'(B)}$.



SIMULTANEOUS DIAGONALIZATION (cont'd.)

- Simultaneous diagonalization is acheived by the transformation

$$\boxed{\boldsymbol{y}' = (\check{\mathbf{E}}_{A/B})^{*T}\boldsymbol{x}}$$

where $\check{\mathbf{E}}_{A/B}$ is the matrix of eigenvectors for the *generalized eigenvalue problem*

$$\boxed{\mathbf{C}_{\boldsymbol{x}(A)}\check{\mathbf{e}}_{A/B} = \check{\lambda}_A \mathbf{C}_{\boldsymbol{x}(B)}\check{\mathbf{e}}_{A/B}}$$

- Covariance matrices $\mathbf{C}_{\boldsymbol{x}(A)}$ and $\mathbf{C}_{\boldsymbol{x}(B)}$ are transformed to

$$\mathbf{C}_{\boldsymbol{y}'(A)} = \check{\mathbf{\Lambda}}_A \qquad \text{and} \qquad \mathbf{C}_{\boldsymbol{y}'(B)} = \mathbf{I}$$

WHITENING TRANSFORMATIONS

- The transformation

$$\boldsymbol{y} = (\mathbf{\Lambda}^{-1/2}\mathbf{E}^{*T})\boldsymbol{x}$$

  which results in the transformed covariance matrix $\mathbf{C}_{\boldsymbol{y}} = \mathbf{I}$ is called a *whitening transformation*.

- All components of the random vector $\boldsymbol{y}$ have unit variance and the concentration ellipsoid is a hypersphere.

MAHALANOBIS TRANSFORMATION

- The *Mahalanobis transformation*

$$\boldsymbol{y} = \mathbf{C}_{\boldsymbol{x}}^{-1/2}\boldsymbol{x} \qquad \text{where} \qquad \mathbf{C}_{\boldsymbol{x}}^{-1/2} = \mathbf{E}\boldsymbol{\Lambda}^{-1/2}\mathbf{E}^{*T}$$

  is another whitening transformation.

- It differs from the previous one in that there is *no net rotation of the coordinate system*.

- The matrix involved in the Mahalanobis transformation is called the *Hermitian square root* of $\mathbf{C}_{\boldsymbol{x}}$ and satisfies

$$\mathbf{C}_{\boldsymbol{x}} = \left(\mathbf{C}_{\boldsymbol{x}}^{1/2}\right)\left(\mathbf{C}_{\boldsymbol{x}}^{1/2}\right)^{*T}$$

DIAGONALIZATION BY TRIANGULAR DECOMPOSITION

$$\boxed{\boldsymbol{x}'' = \mathbf{L}^{-1}\boldsymbol{x} \qquad \Longleftrightarrow \qquad \mathbf{R}_{\boldsymbol{x}''} = \mathbf{L}^{-1}\mathbf{R}_{\boldsymbol{x}}(\mathbf{L}^{-1})^{*T} = \mathbf{D}_L}$$

$\mathbf{L}$ and $\mathbf{D}_L$ are factors in the triangular decomposition

$$\mathbf{R}_{\boldsymbol{x}} = \mathbf{L}\mathbf{D}_L\mathbf{L}^{*T}$$

$\mathbf{L}$ is lower triangular with unit diagonal elements,
$\mathbf{D}_L$ is diagonal.

QR FACTORIZATION

GENERAL FORM

$\mathbf{X} = \mathbf{Q}\mathbf{R}$ where $\mathbf{Q}$ is unitary, $\mathbf{R}$ is upper triangular

$$
\begin{aligned}
\mathbf{X} &= \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} | & & | \\ | & & | \\ \mathbf{q}_1 & \cdots & \mathbf{q}_N \\ | & & | \\ | & & | \end{bmatrix}}_{\mathbf{Q}_1} \underbrace{\begin{bmatrix} | & & | \\ | & & | \\ \mathbf{q}_{N+1} & \cdots & \mathbf{q}_K \\ | & & | \\ | & & | \end{bmatrix}}_{\mathbf{Q}_2} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1N} \\ 0 & r_{22} & \cdots & r_{2N} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & r_{NN} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}
\end{aligned}
$$

QR FOR TRIANGULAR DECOMPOSITION

ESTIMATE FOR CORRELATION MATRIX

$$\hat{\mathrm{R}}_{\boldsymbol{x}} = \frac{1}{K}\mathbf{X}^{*T}\mathbf{X} \qquad (\mathbf{X} \text{ is } K \times N \text{ with } K > N)$$

RELATIONS

$$\hat{\mathbf{D}}_L = \frac{1}{K}\left(\mathrm{diag}\left(\mathbf{R}_1\right)\right)^2 \qquad \hat{\mathbf{L}} = \frac{1}{\sqrt{K}}\mathbf{R}_1^{*T}\hat{\mathbf{D}}_L^{-\frac{1}{2}}$$

TRIANGULAR DECOMPOSITION FORMS

Matrix	Lower-upper decomposition	Upper-lower decomposition
$\mathbf{R}_{\boldsymbol{x}}$	$\mathbf{R}_{\boldsymbol{x}} = \mathbf{L}\mathbf{D}_L\mathbf{L}^{*T}$	$\mathbf{R}_{\boldsymbol{x}} = \mathbf{U}_1\mathbf{D}_U\mathbf{U}_1^{*T}$
$\tilde{\mathbf{R}}_{\boldsymbol{x}}$	$\tilde{\mathbf{R}}_{\boldsymbol{x}} = \tilde{\mathbf{U}}_1\tilde{\mathbf{D}}_U\tilde{\mathbf{U}}_1^{*T}$	$\tilde{\mathbf{R}}_{\boldsymbol{x}} = \tilde{\mathbf{L}}\tilde{\mathbf{D}}_L\tilde{\mathbf{L}}^{*T}$